Q1) Euler implicite car $x(t_{n+1})$.

Q2) Décomposition de Cholesky :

$$\begin{pmatrix} a^2 & ab \\ ab & b^2+c^2 \end{pmatrix}.$$ H : matrice symétrique, définie, posi

$$A = L.U = \begin{pmatrix} \cancel{e}_{11} & 0 \\ \ell_{12} & \cancel{\ell_{22}} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} = \begin{pmatrix} e_{11}u_{11} & e_{11}u_{12} \\ \ell_{12}u_{11} & \ell_{12}u_{12}+\ell_{22}u_{22} \end{pmatrix}$$

$\ell_{11}u_{11} = a^2.$
$\ell_{11}u_{12} = ab.$
$\ell_{12}u_{11} = ab.$
$\ell_{12}u_{12} + \ell_{22}u_{22} = b^2+c^2.$

(cas général).

$A = LU$
$LUX = B.$
$LY = B.$

Cas matrice symétrique :

$$A = L \times {}^tL = \begin{pmatrix} e_{11} & 0 \\ e_{12} & e_{22} \end{pmatrix} \times \begin{pmatrix} e_{11} & e_{12} \\ 0 & e_{22} \end{pmatrix} = \begin{pmatrix} e_{11}^2 & e_{11}e_{12} \\ e_{12}e_{11} & e_{12}^2+e_{22}^2 \end{pmatrix}.$$

L : matrice triangulaire inférieure.

$\Rightarrow \cdot\ e_{11}^2 = a^2.$
$\cdot\ e_{11}e_{12} = ab.$
$\cdot\ e_{12}e_{11} = ab.$
$\cdot\ e_{12}^2 + e_{22}^2 = b^2+c^2.$

$\Rightarrow e_{11} = a.$ $e_{12} = b.$ $e_{22} = c.$ $\Rightarrow L = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.$ $\Rightarrow {}^tL = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$

$\Rightarrow$ Décomposition : $$\begin{pmatrix} a^2 & ab \\ ab & b^2+c^2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

Q3) $\begin{cases} LY = B. \\ {}^tLX = Y. \end{cases}$ $\begin{pmatrix} a^2 & ab \\ ab & b^2+c^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}.$

$(L\,{}^tLX = B).$ $\Rightarrow \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}.$

$\cdot\ LY = B \Rightarrow \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix} \Rightarrow \begin{pmatrix} ay_1 \\ by_1+cy_2 \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}. \Rightarrow \begin{cases} y_1 = b/a \\ y_2 = 1 - b^2/ac \end{cases}$

$\cdot\ {}^tLX = Y \Rightarrow \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b/a \\ 1-b^2/ac \end{pmatrix}. \Rightarrow \begin{pmatrix} ax+by \\ cy \end{pmatrix} = \begin{pmatrix} b/a \\ 1-b^2/ac \end{pmatrix}$

$\Rightarrow \boxed{y = \frac{ac-b^2}{c^2}}$ $\Rightarrow \boxed{x = \frac{b}{a^2} - b\left(\frac{ac-b^2}{ac^2}\right)}.$ $\Rightarrow \boxed{\frac{bc^2 - bac + b^3}{a^2c^2} = x}$